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SOFT AND COLLINEAR BEHAVIOUR OF DUAL AMPLITUDES *

Michelangelo Mangano and Stephen J. Parke
Fermi National Accelerator Laboratory
PO Box 500, Batavia, IL 60510

SUMMARY *The dual representation for multi-gluon matrix elements is used to study the soft and collinear limits of multi-parton scattering. We propose an amplitude to exactly describe one set of helicity amplitudes for two quark - n gluon scattering.*

A new technique was recently proposed to calculate tree-level multi-gluon amplitudes [1,2]. The basic idea is to represent the matrix element for a gluon scattering process in a $SU(N)$ Yang-Mills theory in the following fashion:

$$A_n = \sum_{perm'} tr(\lambda_1\lambda_2\ldots\lambda_n)\, A(1,2,\ldots,n). \tag{1}$$

We will call the functions $A(1,2,\ldots,n)$ *sub-amplitudes*. The matrices λ_a represent the $SU(N)$ algebra in the fundamental representation, and the sum is taken over all the $(n-1)!$ *non-cyclic* permutations of the indices $1,2,\ldots,n$. We will normalise the λ matrices so that $[\lambda_a,\lambda_b] = i\sqrt{2}f_{abc}\lambda_c$ and $tr(\lambda_a\lambda_b) = \delta_{ab}$. The sub-amplitudes satisfy many important properties, as pointed out in [1,2]:

- $A(1,2,\ldots,n)$ is invariant under cyclic permutations of $(1,2,\ldots,n)$.
- $A(1,2,\ldots,n) = (-1)^n A(n,n-1,\ldots,1)$.
- $A(1,2,\ldots,n)$ is gauge invariant, and satisfies the following identity: $\sum' A(1,2,3,\ldots,n) = 0$, where the sum is over the $n-1$ *cyclic* permutations of $(2,3,\ldots,n)$.
- Incoherence to leading order in N:

$$\sum_{colors} |M_n|^2 = N^{n-2}(N^2-1)\sum_{perm'}\{|A(1,2,\ldots,n)|^2 + O(N^{-2})\}. \tag{2}$$

These properties suggest that the sub-amplitudes might be much easier to calculate than the full amplitude, and that the resulting expressions will be remarkably simple. That this is in fact the case was shown in [1,2] through the calculation of the 4,5 and 6 gluon matrix elements.

In [1] the representation (1) was chosen because of its connection with the dual models, or, more precisely, because of the identification of the zero-slope limit of the $SU(N)$ open string with the $SU(N)$ Yang-Mills theory. This analogy suggested that the sub-amplitudes (which in this language should then more properly be referred to as *dual amplitudes*) satisfy a further property in addition to those described above. Namely, the factorisation on the m-particle poles $(p_1 + p_2 + \cdots + p_m = P)$:

$$A(1,2,\ldots,n) \stackrel{P^2\to 0}{\to} A(1,2,\ldots,m,-P)\,\frac{i}{P^2}\,A(P,m+1,\ldots,n). \tag{3}$$

This factorization is a consequence of the general properties of string theories, and is expected to hold for $m \geq 3$. When $m = 2$, equation (3) is not well defined. In fact the amplitude for three on-shell gluons vanishes, and the amplitude among off-shell states is not well defined in string theory as it is formulated today. Then it is not clear what $A(1,2,-P)$ should be if $P^2 \neq 0$. Furthermore, no particular factorization is expected *a priori* when one of the gluons becomes soft. In spite of this, in reference [1] we showed explicitly for the 4,5 and 6 gluon amplitudes that the factorisation holds not only for the three-gluon poles, but also for the two-gluon poles as well as for the soft gluon emission. In addition, by properly regularising the Koba-Nielsen representation of the amplitude when some of the gluons are off-shell it is possible to explicitly prove that this factorisation of the soft and collinear singularities holds for any n [4]:

$$A(1,2,\ldots,n) \stackrel{1\ soft}{\to} E(n,1,2)A(2,3,\ldots,n), \tag{4}$$

$$A(1,2,\ldots,n) \stackrel{1\|2}{\to} \sum_{hel}\frac{1}{\sqrt{2}}V(1,2,-P)\frac{i}{P^2}A(P,3,\ldots,n) \tag{5}$$

$E(n,1,2)$ is the square root of the eikonal factor: $E(n,1^+,2) = g\sqrt{2}(\langle n2\rangle/\langle n1\rangle\langle 12\rangle)$. The superscript + represents the helicity of the gluon, while the symbol $\langle ij\rangle$ is defined in [3,1,2] and satisfies $|\langle ij\rangle|^2 = s_{ij}$. For a soft gluon with negative helicity the vertex is obtained by taking the complex conjugate of $E(n,1^+,2)$. $V(1,2,P)$ is the off-shell three-gluon vertex as derived from the Feynman rules (up to the color factor), and the sum in (5) refers to the two physical helicities of the intermediate gluon. When the gluons 1 and 2 become collinear V reduces to:

$$V(1^-,2^-,P^-) = \frac{-i\sqrt{2}g\langle 12\rangle}{\sqrt{z(1-z)}}, \quad V(1^+,2^-,P^+) = \frac{-i\sqrt{2}gz^2\langle 12\rangle}{\sqrt{z(1-z)}}. \tag{6}$$

The gluon polarisation vectors have been chosen following the prescription developed by Xu *et.al.* in [3]. z is the momentum fraction of gluon 1: $p_1 = zP$, $p_2 = (1-z)P$. The gluons 1 and 2 are incoming, while the gluon with momentum P is outgoing. The vertex V for the other possible helicity configurations can be obtained by complex conjugation and permutations of (6). Equations (4),(5) are very important, because they allow one to approximate multi-gluon amplitudes when some of the gluons are either soft or collinear. The exact knowledge of the six-gluon amplitude allows for instance a very good approximation of the 7- or 8-gluon matrix element when one or two gluons are soft compared

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to the others. Furthermore, by iterating equation (4) and squaring the resulting amplitude one easily recovers the well known expression [5,6]:

$$\sum_{(col,hel)} |A_n|^2 \sim N^{n-2}(N^2-1) \sum_{perm'} \frac{s_{12}^4 + s_{13}^4 + \cdots + s_{1n}^4}{s_{12}s_{23}\cdots s_{n1}} + O(N^{-2}). \tag{7}$$

The advantage of this approach is the possibility to calculate the subleading terms in equation (7). In ref.[7] it was noticed that equation (7), multiplied by a proper weight, actually accounts for more than 90% of the cross section for the six-gluon process in at least 80% of the generated events. In the same paper the conjecture was made that equation (7) represents an even better approximation to the n-gluon cross section for $n > 6$. Iterating equation (5) we get an approximated amplitude that after squaring has the same structure as equation (7), but also contains the gluon-fusion functions [8] corresponding to the collinear emissions. We believe that the inclusion of these contributions (paying proper attention to not double counting some regions of phase space!) may further improve the estimates presented in [7], without increasing the complexity of the numerical calculation.

As a further example of the power of the representation in terms of dual amplitudes, let us derive the matrix element for the collinear emission of a quark-antiquark pair. Let us first substitute equation (5) into equation (1). It is easy to obtain:

$$A_n \to \sum_a \sum_{perm''} i\, tr(\lambda_a \lambda_3 \ldots \lambda_n) f_{a12} V(1,2,P) \frac{i}{P^2} A(P,3,\ldots,n). \tag{8}$$

Now the sum runs over $a = 1,\ldots,N^2-1$ and over *all* the permutations of $(3,4,\ldots,n)$. To extract the amplitude for two collinear quarks and $n-2$ gluons it is now sufficient to substitute the three-gluon vertex $f_{a12}V(1,2,P)$ with a $q\bar{q}g$ vertex $\lambda^a_{12}\bar{u}(1)\gamma_\mu v(2)\epsilon^\mu(P)/\sqrt{2}$. In this way we obtain the following expression for the amplitude:

$$A(2q,(n-2)g) \stackrel{1\|2}{\to} \sum_{perm''} (\lambda_3\ldots\lambda_n)_{ij} \Big\{ \frac{g\,z}{\langle 12\rangle^*} A(P^+,3,\ldots,n) + \frac{g\,(1-z)}{\langle 12\rangle} A(P^-,3,\ldots,n) \Big\}. \tag{9}$$

Once again the sum is over all the permutations of $(3,4,\ldots,n)$ and we have assumed that the quark has positive helicity and momentum fraction z. If all of the emitted gluons but one, say gluon 3, have the same helicity, say positive, then only the contribution proportional to $A(P^-,3^-,\ldots,n^+)$ will survive. If we then use the exact expression for the matrix element $A(P^-,3^-,\ldots,n)$, it is possible to absorb the z factor and to get an expression that makes sense even when the quarks are not collinear:

$$A(q^+,\bar{q}^-;g^-,g^+,\ldots,g^+) = i\langle 13\rangle\langle 23\rangle^3 \sum_{perm''} (\lambda_3\ldots\lambda_n)_{ij} \frac{1}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}. \tag{10}$$

p_1 and p_2 are the momenta of q and $\bar{q}$, respectively, and i and j are their colors. At this stage the identity expressed by equation (10) is just a guess, justified by the fact that by construction it satisfies the required factorization properties for the emission of a soft gluon and collinear pairs. For $n = 4,5$, though, this amplitude gives rise to the exact result [9]. For $n = 6$ the agreement with the exact result [10] is numerical. We speculate that it will be exact even for $n > 6$. For $n \geq 6$, though, it does not describe the full process $(2q,(n-2)g)$, because different helicity configurations are present. It might well be, nevertheless, that the contribution from this helicity configuration will dominate the full $(2q,(n-2)g)$ cross section, in the same way in which the Parke and Taylor expression for $(g^-,g^-,g^+,\ldots)$, equation (7), dominates the n-gluon process. In this way very fast numerical programs might describe with good accuracy the dominant processes responsible for multijet phenomena [11]. To the leading order in N, the square of (10) summed over the colors and over the relevant helicity configurations is:

$$\sum_{hel}\sum_{col} |A(2q,(n-2)g)|^2 = 2N^{n-3}(N^2-1) \cdot \sum_{perm''} \left(\frac{\sum_{a=3}^n s_{1a}s_{2a}(s_{1a}^2+s_{2a}^2)}{s_{12}s_{23}\cdots s_{n1}} + O(N^{-2}) \right). \tag{11}$$

The sum on the right hand side is over all the permutations of $(3,\ldots,n)$ and no average factor has been introduced.

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